differential equations, they can also be solved in the case of practical importance of piecewise-constant functions $a_{ijln}(y_1, y_2, z)$ that model a fibrous material or composite comprised or a periodic system of grains and a material filling the space between them. In this case the following continuity conditions on the grain surface /1, 2/

$$[U_l^{\mu\nu}] = 0, \; [\lambda^{-1} b_{l\delta}^{\mu\nu} n_\delta + b_{l3}^{\mu\nu} n_3] = 0 \qquad (7.6)$$
$$[V_l^{\mu\nu}] = 0, \; [\lambda^{-1} c_{l\delta}^{\mu\nu} n_\delta + c_{l3}^{\mu\nu} n_3] = 0$$

must be appended to the local problems.

Here n_i are components of the vector normal to the contact surface, where we have $A_{i\mu\nu} = 0$ in (3.8) and (4.6).

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VARIATIONAL METHODS IN THREE-DIMENSIONAL PROBLEMS OF NON-STATIONARY INTERACTION OF ELASTIC BODIES WITH FRICTION*

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Three-dimensional contact problems are examined for the interaction between a moving elastic body and an elastic foundation under friction conditions. The desired friction force and slip fields depend on the time. A boundary value problem is formulated in the velocities and is reduced to a parabolic variational inequality. Its difference approximation is proposed and will be used to provide a foundation for formulating the problem in increments. A number of methods is proposed for the numerical solution of the problem. The time behaviour of the solution of the non-stationary problem is investigated. The non-stationarity effects in contact problems with friction are considered first under conditions of body displacement relative to the foundation /1/. Three-dimensional problems formulated in increments of the desired functions were studied in /2/. Quasistatic problems in increments and dynamic problems on the contact between a stamp and elastic solid of finite size were investigated /3/. The method of reducing the non-stationary parabolic problems to sequences of variational problems (in application to viscoplastic flow problems) was used in /4, 5/.

1. Kinematic relationships. Boundary conditions. We examine the motion of an elastic body on an elastic foundation with a plane surface. We consider the velocities of the

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surfaces making contact to be made up of the velocities of the solid and the foundation, considered as absolutely rigid, and additional velocities that occur due to elastic strains in the contact domain. We introduce an $Oxyz$ system of coordinates with origin at the point of tangency between the solid and the foundation in the undeformed state, moving over the foundation with a velocity of this point $\mathbf{V}(V_x, V_y)$. Let $\mathbf{v}^\pm(v_x^\pm, v_y^\pm)$ be the field of tangential velocities of the solid and foundation surfaces in the neighbourhood of the point O without taking account of the elastic strains, and $\mathbf{u}^\pm(u_x^\pm, u_y^\pm)$ are the elastic tangential displacements of points of these surfaces. Then for the total tangential velocities $\mathbf{s}^\pm$ and the slip velocities of the surfaces in the contact domain $\mathbf{s}$ we will have

$$\mathbf{s}^\pm = \mathbf{v}^\pm + \frac{d\mathbf{u}^\pm}{dt} = \mathbf{v}^\pm + \mathbf{u}^{\cdot\pm} + \mathbf{v}^\pm \operatorname{grad} \mathbf{u}^\pm \tag{1.1}$$

$$\mathbf{s} = \mathbf{v} + \mathbf{u}^{\cdot} - \mathbf{V} \operatorname{grad} \mathbf{u}, \quad \mathbf{v} = \mathbf{v}^+ - \mathbf{v}^-, \quad \mathbf{u} = \mathbf{u}^+ - \mathbf{u}^-, \quad \mathbf{u}^{\cdot} = \frac{d\mathbf{u}}{dt} \tag{1.2}$$

Assuming henceforth that $\mathbf{v}$ is a function of x, y, t, we consider for simplicity that the velocity $\mathbf{v}$ does not change with time.

The three components on the right-hand side of the first equality in (1.2) define the "hard" local component (because of non-stationarity of the process) and the transfer component (because of coordinate system motion) of the slip velocity. Together with the general case of taking account of all the components mentioned (for example, during roll with slip) we will also consider the case of slow body motion when the transfer velocities are much less than the remaining slip components and cannot be taken into account. This occurs during a displacement with respect to the foundation of a body initially at rest when its displacement velocity is of the order of the elastic strain rates.

The boundary conditions, which are referred to the $z = 0$ plane because the contact is local, will have the form

$$w = w^+ - w^- > \delta - f^+ = F, \ \sigma_{zz} = |\tau| = 0 \tag{1.3}$$
$$w = F, \ \sigma_{zz} \leqslant 0$$

$$|\mathbf{s}| = 0, \ |\boldsymbol{\tau}| \leqslant -\rho(\sigma_{zz})\sigma_{zz} \tag{1.4}$$
$$|\mathbf{s}| > 0, \quad \boldsymbol{\tau} = -\rho(\sigma_{zz})\sigma_{zz}\mathbf{s}/|\mathbf{s}|$$

where $w^\pm$ are elastic normal displacements of the body and foundation surfaces, σ_{zz} are normal stresses, $\boldsymbol{\tau}(\tau_{xz}, \tau_{yz})$ tangential stresses on the surface, f^+ is the function giving the body surface, and δ is the normal closure between the body and the foundation. We consider the function F to be independent of time.

Relationships (1.3) describe the free body and foundation surfaces and their contact domain, while relationships (1.4) describe the friction conditions in the contact domain (in the adhesion and slip domains). We confine ourselves below to cases when problem (1.3) of determining the contact domain and its normal stresses is separated from problem (1.4) of finding the friction forces. In particular, for identical elastic constants of the body and the foundation this will be either when one of them is incompressible, and the other is absolutely rigid, or both are incompressible. We shall later investigate problem (1.4) under the assumption that σ_{zz} are determined in the contact domain E. Methods of finding them and examples of calculations are given in /6-8/, say.

Using the same expressions for the surface displacements as for two half-spaces $Z < 0$ and $Z > 0$, we will have, from (1.2)

$$\mathbf{s} = \mathbf{v} - B(\boldsymbol{\tau}^{\cdot}) + B^*(\boldsymbol{\tau}), \quad B^*(\cdot) = -V_x \frac{\partial}{\partial x} B(\cdot) - V_y \frac{\partial}{\partial y} B(\cdot) \tag{1.5}$$

where B is an integral operator with the kernel

$$\frac{1}{\pi G R} \begin{Vmatrix} 1 - \nu \sin^2\theta & \nu \sin\theta\cos\theta \\ \nu \sin\theta\cos\theta & 1 - \nu\cos^2\theta \end{Vmatrix}$$
$$G^{-1} = {}^1/_2(G_+^{-1} + G_-^{-1}), \quad \nu = {}^1/_2 G(\nu_+ G_+^{-1} + \nu_- G_-^{-1})$$
$$R = \sqrt{(x-x')^2 + (y-y')^2}, \quad \cos\theta = x - x'/R, \ \sin\theta = y - y'/R$$

$G_{+(-)}$, $\nu_{+(-)}$ are the shear moduli and Poisson's ratios of the body and the foundation.

Thus, a non-stationary boundary value problem (1.4), (1.5) has been formulated to determine $\boldsymbol{\tau}(x, y, t)$. The following reasoning is used below to solve it. System (1.4) (taking (1.5) into account) can be written in the form of a single inequality

$$\boldsymbol{\tau}_0 \mathbf{s}(\boldsymbol{\tau}_0) \geqslant \boldsymbol{\tau}\mathbf{s}(\boldsymbol{\tau}_0) \tag{1.6}$$

which holds for any x, y, t and a selected $\boldsymbol{\tau}_0$, the solution of problem (1.4), among the functions

τ satisfying the inequality $|\tau| \leqslant -\rho(\sigma_{zz})\sigma_{zz}$. Indeed, the left and right sides of (1.6) are zero in the adhesion domain, the left side agrees with the quantity $-\rho(\sigma_{zz})\sigma_{zz}|s(\tau_0)|$ in the slip domain, while the right side does not exceed this quantity. The quantity τ_0 can be extracted from the family τ also in the form of the relationship

$$\tau_0 s(\tau_0) = \max_{|\tau| \leqslant -\rho(\sigma_{zz})\sigma_{zz}} \tau s(\tau) \tag{1.7}$$

We emphasize that unlike (1.7), the right side of (1.6) is not determined by the varied function τ. Consequently, the difference between the right and left sides of (1.6) does not determine the increment of a certain functional. However, in some cases it can determine part of the increment (linear, say) of a functional and then in the presence of its convexity this is sufficient to give an equivalent extremal form to condition (1.6). Therefore, on the basis of (1.6) and (1.7) different variational formulations can be obtained for the boundary value problem, each of which possesses certain advantages.

2. Reduction to a sequence of stationary inequalities. Foundation of a formulation in increments. We consider the problem in an interval $t \in [0, T]$. We introduce the spaces V and V' of the function $f(x, y, t)$ with the normal defined by the equalities

$$\|f\|^2_V = \int_0^T \int_E f^2\,dx\,dy\,dt, \quad \|f\|^2_{V'} = \|f\|^2_V + \int_0^T \int_E f^{\cdot 2}\,dx\,dy\,dt \tag{2.1}$$

We will assume that $\mathbf{f}(f_x, f_y) \in V$ if $f_{x(y)} \in V$. Let K be the set $\mathbf{f}$ such that $|\mathbf{f}| \leqslant -\rho(\sigma_{zz})\sigma_{zz}$.

Theorem 1. Let $\mathbf{v} \in V$, $\rho(\sigma_{zz})\sigma_{zz} \in L_2(E)$, then the solution of problem (1.4), (1.5) under the initial conditions $\tau(0) = \tau^0$ is equivalent to finding the function $\tau_0 \in K \cap V'$ which satisfies, for any $\tau \in K \cap V'$, the evolutionary parabolic inequality

$$\int_0^r \int_E [B(\tau_0^{\cdot}) - B^*(\tau_0) - \mathbf{v}](\tau - \tau_0)\,dx\,dy\,dt \geqslant 0 \tag{2.2}$$
$$\tau_0(0) = \tau^0, \quad r \in [0, T]$$

Proof. The B and B^* are, respectively, an operator with a weak singularity and a singular integral operator. As is well-known /9/, they act from $L_2(E)$ into $L_2(E)$. Therefore, the left side of (2.2) is defined.

If the function τ_0 satisfies conditions (1.4) $\forall t \in [0, T]$ the following relationships are obviously satisfied:

$$G(\tau) \leqslant I(\tau_0), \; G(\tau_0) = I(\tau_0)$$
$$\left(G(\tau) = \int_0^r \int_E s(\tau_0^{\cdot})\,\tau\,dx\,dy\,dt, \quad I(\tau_0) = -\int_0^r \int_E \rho(\sigma_{zz})\sigma_{zz}|s(\tau_0^{\cdot})|\,dx\,dy\,dt\right)$$

Hence, taking (1.5) into account we obtain the inequality (2.2).

To derive the boundary conditions from (2.2), we write (2.2) in the form $G(\tau) \leqslant G(\tau_0)$. As in /10/, it can be shown that

$$\sup_{\tau \in K \cap V'} G(\tau) = I(\tau_0)$$

from which these results

$$G(\tau_0) = I(\tau_0), \quad \int_0^r \int_E [s(\tau_0^{\cdot})\,\tau_0 + \rho(\sigma_{zz})\sigma_{zz}|s(\tau_0^{\cdot})|]\,dx\,dy\,dt = 0$$

Since $\tau_0 \in K$, we obtain that for almost all t and x, y the equality

$$s(\tau_0^{\cdot})\,\tau_0 = -\rho(\sigma_{zz})\sigma_{zz}|s(\tau_0^{\cdot})|$$

equivalent to the boundary conditions (1.4), is satisfied.

We will now consider methods of solving (2.2). We will reduce the evolutionary inequality (2.2) to a sequence of stationary elliptic inequalities. We separate the segment $[0, T]$ into equal intervals $\Delta t = T/N$. We examine the sequence of variational inequalities

$$\int_E [B(\tau_N^{k+1} - \tau_N^k) - \Delta t B^*(\tau_N^{k+1}) - \mathbf{v}^k \Delta t](\tau - \tau_N^{k+1})\,dx\,dy \geqslant 0 \tag{2.3}$$
$$\tau \in K, \quad 0 \leqslant k \leqslant N-1, \quad \tau_N^0 = \tau^0, \quad \mathbf{v}^k = \frac{1}{\Delta t}\int_{k\Delta t}^{(k+1)\Delta t} \mathbf{v}(t)\,dt$$

which are obtained by an implicit difference approximation /11/ of inequalities (2.2).

Theorem 2. For any fixed k and N the solution of (2.3) exists and is unique in $L_2(E)$.

Proof. The kernel of the operator B^* is skew-symmetric relative to the variables x, y and x', y'; consequently for any τ_1, τ_2 the following equality holds

$$\int_E \tau_1 B^*(\tau_2)\, dx\, dy = -\int_E \tau_2 B^*(\tau_1)\, dx\, dy \tag{2.4}$$

Using the different structure of the kernel of the operator B and applying the Fourier transform F_ξ with parameter $\xi(\xi_1, \xi_2)$ taking the relationships

$$F_\xi\left(\frac{\sin^i\theta\cos^j\theta}{2\pi R}\right) = \frac{\xi_1{}^i\xi_2{}^j}{|\xi|^3}, \quad 0 \leqslant i, \quad j \leqslant 2, \quad i+j=2$$

into account, we obtain the inequality

$$\int_E \tau B(\tau)\, dx\, dy \geqslant \frac{1-\nu}{2\pi^2 G}\int_\xi \frac{|F_\xi(\tau)|^2}{|\xi|}\, d\xi_1\, d\xi_2 \tag{2.5}$$

The monotonicity of the operator $B - \Delta t B^*$ results from (2.4) and (2.5). This operator is continuous in $L_2(E)$ /9/. Moreover, $\mathrm{v}^k \in L_2(E)$ for $\mathrm{v} \in V$, the set K is convex and closed in $L_2(E)$. Satisfaction of these requirements ensures the existence of the solution.

An assumption about the possibility of two solutions τ_1, τ_2 and their substitution into (2.3) will result in the inequality

$$W(\tau_1 - \tau_2) = \frac{1}{2}\int_E B(\tau_1 - \tau_2)(\tau_1 - \tau_2)\, dx\, dy \leqslant 0 \tag{2.6}$$

To prove the impossibility of (2.6) it is sufficient to show by virtue of (2.5) that

$$\int_\xi \frac{|F_\xi(\tau_1) - F_\xi(\tau_2)|^2}{|\xi|}\, d\xi_1\, d\xi_2 > 0 \tag{2.7}$$

But if $\|\tau_1 - \tau_2\|_{L_2(E)} > 0$, then by virtue of the Parseval equality

$$\int_\xi |F_\xi(\tau_1) - F_\xi(\tau_2)|^2 d\xi_1\, d\xi_2 > 0$$

from which (2.7) and the uniqueness of the solution of (2.3) result.

Now, let us integrate by parts on the left side of (2.2) and let us convert the inequality to the equivalent form

$$\int_0^r \int_E [B(\tau^{\cdot}) - B^*(\tau_0) - \mathrm{v}](\tau - \tau_0)\, dx\, dy\, dt - \tag{2.8}$$
$$\frac{1}{2}\int_E [B(\tau - \tau_0)(\tau - \tau_0)\, dx\, dy]_{t=0}^{t=r}, \quad \forall\tau \in K \cap V', \quad \forall r \in [0, T]$$

In the terminology of /12, (2.2) and (2.8) are called, respectively, the strong and weak forms of the variational parabolic inequality. Since it is assumed in the examination of (2.2) that $\tau \in V'$ (this is a narrower class of allowable functions than in /12/), then the inequality (2.8) is equivalent to (2.2) /13/.

We construct the function $\tau_N(t) \in V$ by means of $\tau_N{}^k$, the solutions of (2.3), by defining it by the conditions

$$\tau_N(t) = \tau_N{}^k \text{ for } t \in [k\Delta t, (k+1)\Delta t], \quad 0 \leqslant k \leqslant N-1$$

Theorem 3. As $N \to \infty$ $(\Delta t \to 0)$ weakly in V the functions τ_N converge to the solution of (2.8).

Proof. The sequence $\tau_N(t)$ is bounded since it belongs to the bounded set K. Therefore, a weakly convergent sequence in V can be extracted from it. We retain the notation τ_N for it.

If it is shown that the limit function τ^* satisfies inequality (2.8) and this solution is unique, then the theorem will be proved since all the other sequences τ_N should also converge to τ^*. To prove uniqueness it is sufficient to write (2.8) twice for $\tau_0 = \tau_1$, $\tau = \tau_2$; $\tau_0 = \tau_2$, $\tau = \tau_1$ (τ_i are the assumed solutions) and then combine the results. Finally, we obtain that inequality (2.6) should be satisfied for all r. Consequently, there cannot be two solutions of (2.8).

We will now prove that τ^* satisfies inequality (2.8). We represent the variable function $\tau \in K \cap V'$ in the form of the limit of the sequence $\tau_m \in C^1([0, T], L_2(E))$ such that $\tau_m \to \tau$, $\tau_m{}^{\cdot} \to \tau^{\cdot}$

in V.

We will show that (2.3) is satisfied for $\tau_0 = \tau^*, \tau = \tau_m$ and then we pass to the limit as $m \to \infty$. To do this we introduce the piecewise-constant function $\tau_c = \tau_m(k\Delta t), t \in [k\Delta t, (k+1)\Delta t]$, $1 \leqslant k \leqslant N-1$ and the piecewise-linear function

$$\tau_l = \tau_m(k\Delta t) + t\{\tau_m[(k+1)\Delta t] - \tau_m(k\Delta t)\}/\Delta t, \quad t \in [k\Delta t, (k+1)\Delta t]$$

Substituting τ_N, τ_c, τ_l in place of τ_0, τ, and $\tau^{\cdot}$, respectively, in the left side of (2.8) (denoted by I_{mn}) below) we obtain

$$I_{mn} = \sum_{k=0}^{N-1} \int_E \{B\{\tau_m[(k+1)\Delta t] - \tau_m(k\Delta t)\} - \Delta t[\mathbf{v}^k + B^*(\tau_N^{k+1})]\}$$
$$\{\tau_m[(k+1)\Delta t] - \tau_N^{k+1}\}\,dx\,dy - \frac{1}{2}\int_E B[(\tau_m(N\Delta t) - \tau_N{}^N)]$$
$$[\tau_m(N\Delta t) - \tau_N{}^N]\,dx\,dy + \frac{1}{2}\int_E B[\tau_m(0) - \tau_N(0)][\tau_m(0) - \tau_N(0)]\,dx\,dy$$

Now, if (as in /12/, Ch.4), we sum the left side of (2.3) over k and use the identity $(\mathbf{a}-\mathbf{b}, \mathbf{a}) = {}^1/_2|\mathbf{a}|^2 - {}^1/_2|\mathbf{b}|^2 - {}^1/_2|\mathbf{a}-\mathbf{b}|^2$, we then obtain that $I_{mn} \geqslant 0$. Thus, the validity of the inequality (2.8) is established for fixed m, N.

We first pass to the limit as $N \to \infty$ $(\Delta t \to 0)$. By virtue of (2.4) we will have

$$\int_0^T \int_E B^*(\tau^N)(\tau_m - \tau_N)\,dx\,dy\,dt = \int_0^T \int_E B^*(\tau_N)\tau_m\,dx\,dy\,dt \to$$
$$\int_0^T \int_E B^*(\tau^*)\tau_m\,dx\,dy\,dt \text{ for } \tau_N \to \tau^* \text{ weakly}$$

The passage to the limit in the N remaining components of the left side of (2.8) is performed taking monotonicity of the operator B as well as the fact that $\tau_c \to \tau_m^{\cdot}$, $\tau_l^{\cdot} \to \tau_m^{\cdot}$, $\mathbf{v} \xrightarrow{k} \mathbf{v}$, $\tau_N \to \tau^*$ (weakly) into account. After this, we pass to the limit over k on the left side of (2.8) by taking account of the properties of the approximation of τ by the functions τ_m.

Remark. The proof presented for Theorem 3 simultaneously establishes the existence of the solution of the variational inequality (2.8) that belongs to the space V.

The original boundary value problem formulated in velocities is equivalent to the evolutionary inequality (2.2). The stationary inequality (2.3) can be set in correspondence with the equivalent boundary value problem formulated in increments of the desired functions. This is obtained from (1.4) and (1.5) by replacing the function $\mathbf{s}(t)$ by

$$\Delta \mathbf{s}^{k+1} = \mathbf{v}^k \Delta t - B(\tau^{k+1} - \tau^k) + \Delta t B^*(\tau^{k+1}) \tag{2.9}$$

The proof of the equivalence of (2.3) and the boundary value problem in increments is analogous to the proof of Theorem 1. Theorem 3 is the foundation for using the formulation of the problem in increments.

3. Reduction to the solution of a sequence of minimization problems. Direct methods of solution /11, 14/ have been actively developed for stationary variational inequalities with a monotonic operator of the type (2.3). However, experience with their practical utilization in solving three-dimensional boundary value problems with conditions in the form of inequalities still gives way to that accumulated in the realization of methods based on the minimization of functionals.

We will now construct such methods for the problems under consideration. In conformity with one of the means noted in Sect.2, we will try to select the functional for which inequality (2.3) is the sufficient condition for an extremum. The operator B^* is non-symmetric, and consequently, because of the presence of the term $\Delta t B^*$ in (2.3), such a functional does not exist in the general case. However, if the transfer velocities can be neglected, the following theorem holds.

Theorem 4. In the case of slow body motions, the solution of the boundary value problem reduces to a sequence of minimization problems for the smooth (quadratic) functionals

$$\min_{|\tau_N^{k+1}| \leqslant -\rho(\sigma_{zz})\sigma_{zz}} \int_E \left[\frac{1}{2} B(\tau_N^{k+1})\tau_N^{k+1} - B(\tau_N{}^k)\tau_N^{k+1} - \mathbf{v}^k \Delta t \tau_N^{k+1}\right] dx\,dy, \quad \tau_N{}^\circ = \tau^\circ \tag{3.1}$$

Proof. In the case under consideration (in the absence of the operator B^*) inequality (2.3) is the condition for positivity of the linear part of the increment of a quadratic functional. As is well-known /12/, the equivalence of (2.3) and (3.1) holds here (taking account of the symmetry of the operator B). In the same sense as in Theorem 3, convergence of the solution of (3.1) to the solution of the original problem occurs.

Remark. The method of gradient projection /15/ using the procedure of step selection proposed by Fedorenko has shown high efficiency in the numerical solution of problem (3.1).

Passage to the sequence of minimization problems can be realized even manually by using an extremal relationship of the type (1.7) and holds for the general case of body kinematics.

Theorem 5. The solution of the boundary value problem reduces to a sequence of minimization problems of non-smooth functionals of the form

$$\min_{|\tau_N^{k+1}|\leqslant -\rho(\sigma_{zz})\sigma_{zz}} \int_E [-\rho(\sigma_{zz})\sigma_{zz}|\Delta s^{k+1}| - \tau^{k+1}\Delta s^{k+1}]\,dx\,dy, \quad \tau_N{}^\circ = \tau^\circ \tag{3.2}$$

where $\Delta s^{k+1}(\tau_N^{k+1}, \tau_N^k)$ is defined by (2.9).

Proof. We consider the boundary value problem (1.4), (2.9) formulated in increments. It is seen from (2.9) that Δs^{k+1} is the result of applying the operator $B - \Delta t B^*$, which is the sum of symmetric and skew-symmetric operators, to the desired function τ_N^{k+1}. An analogous boundary value problem in which either a symmetric or skew-symmetric operator was concerned was examined in /10/, where its equivalence to the minimization problem (3.2) was proved. It turns out that the method of proof in /10/ carries over completely to the case considered here of the presence of the operator $B - \Delta t B^*$. The convergence of the solution of the problem (3.2) as $N \to \infty$ $(\Delta t \to 0)$ results from Theorem 3.

Remark. A special method* (*Fedorenko, R.P., A method for the numerical solution of three-dimensional contact problems of roll with slip and adhesion. Preprint No.158, Inst. Applied Math., USSR Academy of Sciences, Moscow, 1979.) based on combining the gradient projection and linear programming methods has been developed for the numerical realization of problems of the type (3.2).

4. Qualitative properties of the solution.

We will examine the formulation of the problem in increments. We call the solution τ^0 of problem (1.4), (2.9) with initial conditions stable if $\forall \varepsilon\, \exists \delta$, that from $\|\tau^\circ - \tau_{in}\| < \delta$ the following inequalities result

$$\|\tau^k(\tau^\circ) - \tau^k(\tau_{in})\| < \varepsilon, \quad \forall k = 1, 2, \ldots \tag{4.1}$$

Here $\tau^k(\tau_{in})$ is the solution with initial conditions τ_{in}.

As is well-known, the operator B acts from the space $H^0_{-1/2}(E)$ into the space $H_{1/2}(E)$ and is hence bounded /9/. Taking this as well as inequality (2.5) into account, we conclude that the estimates

$$c_1\|\tau\|^2_{H^0_{-1/2}(E)} \leqslant W(\tau) = \frac{1}{2}\int_E B(\tau)\tau\,dx\,dy \leqslant c_2\|\tau\|^2_{H^0_{-1/2}(E)} \tag{4.2}$$

hold for any τ.

Furthermore, to analyse the stability of the solution we use the norm of the space $H_{-1/2}{}^0(E)$, which is the energetic space of the operator B, as follows from (4.2).

Theorem 6. Let $\mathbf{v}^k = \mathbf{v}(x, y)$, $\forall k$, then any solution of problem (1.4), (2.9) is stable.

Proof. We will show that the inequality $W^k = W(\tau^k(\tau^\circ) - \tau^k(\tau_{in}))$ does not increase as k increases. Indeed, by the positive-definiteness of the operator B we have

$$W^{k+1} - W^k = L = \int_E B(\tau^{k+1}(\tau^\circ) - \tau^{k+1}(\tau_{in}) - \tau^k(\tau^\circ) + \tau^k(\tau_{in}))(\tau^{k+1}(\tau^\circ) - \tau^{k+1}(\tau_{in}))\,dx\,dy \tag{4.3}$$

On the other hand, if we write inequality (2.3) twice, for $\tau_N^{k+1} = \tau^{k+1}(\tau^\circ)$, $\tau = \tau^{k+1}(\tau_{in})$ and $\tau_N^{k+1} = \tau^{k+1}(\tau_{in})$, $\tau = \tau^{k+1}(\tau^\circ)$, and then combine the results, we obtain that $L \leqslant 0$. Therefore, we will have the sequence of inequalities

$$c_1\|\tau^k(\tau^\circ) - \tau^k(\tau_{in})\|^2 \leqslant W^k \leqslant W^\circ \leqslant c_2\|\tau^\circ - \tau_{in}\|^2$$

from which the stability of the solution obviously results.

Remark. The stability of any solution of the continuous problem (1.4), (1.5) can also be proved by using a continuous function of the time $W(t)$ (the analgoue of the Lyapunov function).

We will examine the case of slow body motions further. We consider the following integral characteristics of the solution of a problem on simultaneous shear and rotation of the body $(v_x = v_x{}^\circ - \omega y,\ v_y = v_y{}^\circ + \omega x)$ for the interval $t \in [k\Delta t, (k+1)\Delta t]$:

$$A^k = \frac{1}{2}\int_E \mathbf{v}\Delta t(\tau^{k+1} + \tau^k)\,dx\,dy, \quad T^k_{x(y)} = \frac{1}{2}\int_E (\tau^{k+1}_{x(y)s} + \tau^k_{x(y)s})\,dx\,dy$$

$$M^k = \frac{1}{2}\int_E [(\tau_{yz}^{k+1} + \tau_{yz}^k)\, x - (\tau_{xz}^{k+1} + \tau_{xz}^k)\, y]\, dx\, dy,$$

$$T^k = \frac{T_x^k v_x^\circ + T_y^\circ v_y^\circ}{\sqrt{(v_x^\circ)^2 + (v_y^\circ)^2}}$$

which are, respectively, the work of the friction forces on rigid slips, the projections of the total friction force acting on the body, on the coordinate axes, the rotational moment of the friction forces around the body axis, and the projection of the total friction force in the shear direction.

Theorem 7. The work of the friction force A^k increases with time (as k increases).

For the proof we consider inequality (2.3) in the absence of transfer velocities. We write it twice, for $k = n,\ \tau = \tau^n;\ k = n + 1,\ \tau = \tau^{n+1}$, and we then combine the results. We will therefore have

$$\frac{1}{2}\int_E \nu\Delta t\, [\tau^{k+2} + \tau^{k+1} - (\tau^{k+1} + \tau^k)]\, dx\, dy \geqslant W(\tau^{k+1} - \tau^k) + W(\tau^{k+2} - \tau^{k+1})$$

Hence $A^{n+1} - A^n \geqslant 0$.

Corollary 1°. Let $\omega = 0$. Then the projection of the friction force in the shear direction increases with time.

2°. Let $\omega = 0,\ v_y^\circ = 0,\ v_x^\circ > 0\ (v_x^\circ = 0,\ v_y^\circ > 0)$. Then the projection of the force $T_x^k\ (T_y^k)$ on the $x\ (y)$ axis increases with time.

3°. Let $v_x^\circ = v_y^\circ = 0,\ \omega > 0$. Then the moment of the friction force M^k increases with time.

These assertions result from the fact that the equalities

$$T^k = A^k / [\sqrt{(v_x^\circ)^2 + (v_y^\circ)^2}\, \Delta t], \quad T_{x(y)}^k / (\overset{\circ}{v}_{x(y)} \Delta t), \quad M^k = A^k / (\omega \Delta t)$$

hold under the conditions specified in them.

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